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1995 J. Phys. A: Math. Gen. 28 975

(http://iopscience.iop.org/0305-4470/28/4/021)

J. Phys. A: Math. Gen. 28 (1995) 975–983. Printed in the UK

The C-matrix and the reality classification of the representations of the homogeneous Lorentz group: III. Irreducible representations of the orthochronous and homogeneous Lorentz groups

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Received 21 March 1994, in final form 9 September 1994

Abstract. The bilinear metrics and the C-matrices admitted by the irreducible representations (irreps) of the orthochronous and the homogeneous Lorentz groups are determined and the irreps are classified into three reality types.

1. Introduction

It is well known (see, for example, Gelfand *et al* (1963) or Barut (1964)) that the orthochronous Lorentz group (OLG) is obtained from the orthochronous proper Lorentz group (OPLG) SO(3, 1) by adding the space-reflection transformation $s = \mathrm{diag}(-1, -1, -1, 1)$ and all possible products of the form sg', where $g' \in \mathrm{SO}(3,1)$, to SO(3, 1). Gelfand *et al* (1963) have proved the following two theorems which completely determine all the irreps of the OLG.

Theorem 1. Any self-conjugate irrep $g' \mapsto \mathbf{D}(g')$ of SO(3, 1) (i.e. an irrep for which either $j_0 = 0$ or $c = 0$) may be used to produce two non-equivalent (inequivalent) irreps of the OLG acting in the same space as the irrep $g' \mapsto \mathbf{D}(g')$, as follows

(i) $\quad g' \mapsto \mathbf{D}(g') \qquad s \mapsto +\mathbf{S}_1 \qquad sg' \mapsto +\mathbf{S}_1\mathbf{D}(g') \qquad \forall g' \in \mathrm{SO}(3,1)$

(ii) $\quad g' \mapsto \mathbf{D}(g') \qquad s \mapsto -\mathbf{S}_1 \qquad sg' \mapsto -\mathbf{S}_1\mathbf{D}(g') \qquad \forall g' \in \mathrm{SO}(3,1).$

The operator $\mathbf{S}_1$, representing the space reflection element s, has, in the canonical Gelfand–Naimark (GN) basis of the irrep $g' \mapsto \mathbf{D}(g')$ of SO(3, 1), the matrix elements‡

$$S_1(j, m; j', m') = (-1)^{j-j_0}\delta_{j',j}\delta_{m',m}. \tag{1}$$

It is readily checked that $\mathbf{S}_1$ possesses the following properties:

$$\mathbf{S}_1 = \mathbf{S}_1^* = \tilde{\mathbf{S}}_1 = \mathbf{S}_1^{-1} \qquad \mathbf{S}_1\mathbf{S}_1^\dagger = \mathbf{E} \tag{2}$$

where the symbols $*$, $\sim$ and $\dagger$ denote, respectively, the complex conjugate, the matrix-transpose and the adjoint (transposed complex conjugate) of the matrix $\mathbf{S}$, and $\mathbf{E}$ is a unit matrix of appropriate dimension.

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‡ For notations and conventions regarding the irreps $\mathbf{D}(j_0, c)$ of SO(3, 1), see Gopala Rao *et al* (1994a).

Theorem 2. The pair of non-equivalent irreps $\mathbf{D}(j_0, c)$ and $\mathbf{D}(j_0, -c)$ with neither j_0 nor c equal to zero (called mutually conjugate irreps of SO(3, 1)), acting in the carrier spaces $B(j_0, c)$ and $B(j_0, -c)$, respectively, may be used to produce an irrep of the OLG acting in the linear sum space $B(j_0, c) \oplus B(j_0, -c)$ as follows

$$g' \mapsto \mathbf{D}(g') \equiv \left[\begin{array}{c|c} \mathbf{D}(j_0, c; g') & 0 \\ \hline 0 & \mathbf{D}(j_0, -c; g') \end{array}\right] \qquad j_0 \neq 0, c \neq 0$$

$$s \mapsto \mathbf{S} \equiv \left[\begin{array}{c|c} 0 & \mathbf{S}_1 \\ \hline \mathbf{S}_1 & 0 \end{array}\right] \qquad sg' \mapsto \mathbf{SD}(g') \qquad \forall g' \in \mathrm{SO}(3, 1).$$

Here, $\mathbf{S}_1$ is the matrix already given in equation (1) and the obvious notation $\mathbf{D}(j_0, c; g')$ stands for the matrix representative of the element $g' \in \mathrm{SO}(3, 1)$ in the $\mathbf{D}(j_0, c)$ irrep. (Note that the two irreps $\mathbf{D}(j_0, c)$ and $\mathbf{D}(j_0, -c)$ both lead to the same matrix $\mathbf{S}_1$ by equation (1); this would follow from the fact that the two carrier spaces $\mathbf{B}(j_0, c)$ and $\mathbf{B}(j_0, -c)$ are essentially the same and, hence, may be completely identified with each other (see also Srinivasa Rao *et al* 1983).)

Next, we may recall (Gelfand *et al* 1963, Barut 1964) that the homogeneous Lorentz group† (HLG) is obtained by adding all products of the form tg' to the OLG, where $g' \in$ OLG and t is the time reflection element $t = \mathrm{diag}(1, 1, 1, -1)$. Thus SO(3, 1) $\subset$ OLG $\subset$ HLG and hence elements of the form g, sg, tg, and jg, where $j = st = ts$ is the total reflection element and $g \in \mathrm{SO}(3, 1)$, exhaust the HLG. The identity element e and the three reflections s, t and j together form a finite abelian group of order 4 called the group of reflections, characterized by the following group multiplication table:

$$st = ts = j \qquad sj = js = t \qquad tj = jt = s \qquad s^2 = t^2 = j^2 = e. \tag{3}$$

As the group of reflections is a subgroup of the HLG, it is clear that every representation of the HLG also automatically generates a corresponding representation of the group of reflections. A representation of the HLG which leads to a unique or single-valued representation of the group of reflections is called a unique representation (Gelfand *et al* 1963) of the HLG, and, in this case, the operator representatives $\mathbf{S}, \mathbf{T}$ and $\mathbf{J}$ of s, t and j, respectively, commute with one another. In contrast, there exist representations of the HLG which lead to two-valued representations for the group of reflections and, consequently, such representations are called two-valued representations of the HLG (Gelfand *et al* 1963). In the case of the two-valued representations of the HLG, one can show (Gelfand *et al* 1963) that the operators $\mathbf{S}, \mathbf{T}$ and $\mathbf{J}$ anti-commute with one another. Gelfand *et al* (1963) have shown that all the unique irreps of the HLG may be obtained by extending the representations of the subgroup SO(3, 1) as follows‡.

Theorem 3. Any self-conjugate irrep $g' \mapsto \mathbf{D}(g')$ of SO(3, 1) (see theorem 1) may be used to produce two non-equivalent unique irreps of the HLG as follows

(i) $\qquad g' \mapsto \mathbf{D}(g') \qquad s \mapsto \mathbf{S}_1 \qquad t \mapsto \mathbf{S}_1 \qquad j \mapsto \mathbf{E} \qquad sg' \mapsto \mathbf{S}_1\mathbf{D}(g')$

† Note that Gelfand *et al* (1963) call the OPLG (SO(3, 1)), the OLG and the HLG, respectively, as the proper Lorentz group, the complete Lorentz group and the general Lorentz group.

‡ Note that Gelfand *et al* (1963) actually obtain the irreps of the HLG by a process of extension of the irreps of the subgroup OLG. However, since all the irreps of the OLG, in turn, are obtained as extensions of the representations of SO(3, 1) from theorems 1 and 2, we have restated theorems 3–6 of Gelfand *et al* (1963) as prescriptions for obtaining the irreps of the HLG directly from the representations of SO(3, 1).

$$tg' \mapsto \mathbf{S}_1\mathbf{D}(g') \qquad jg' \mapsto \mathbf{D}(g') \qquad \forall g' \in \mathrm{SO}(3,1)$$

(ii) $$g' \mapsto \mathbf{D}(g') \qquad s \mapsto \mathbf{S}_1 \qquad t \mapsto -\mathbf{S}_1 \qquad j \mapsto -\mathbf{E} \qquad sg' \mapsto \mathbf{S}_1\mathbf{D}(g')$$
$$tg' \mapsto -\mathbf{S}_1\mathbf{D}(g') \qquad jg' \mapsto -\mathbf{D}(g') \qquad \forall g' \in \mathrm{SO}(3,1).$$

Theorem 4. Any pair of non-equivalent mutually conjugate irreps of SO(3, 1) (see theorem 2) may be used to produce two non-equivalent unique irreps of the HLG as follows:

(i) $$g' \mapsto \mathbf{D}(g') \equiv \left[\begin{array}{c|c} \mathbf{D}(j_0, c; g') & 0 \\ \hline 0 & \mathbf{D}(j_0, -c; g') \end{array}\right] \qquad j_0 \neq 0 \qquad c \neq 0$$
$$s \mapsto \mathbf{S} \equiv \left[\begin{array}{c|c} 0 & \mathbf{S}_1 \\ \hline \mathbf{S}_1 & 0 \end{array}\right]$$
$$t \mapsto \mathbf{S} \qquad j \mapsto \mathbf{E} \qquad sg' \mapsto \mathbf{SD}(g') \qquad tg' \mapsto \mathbf{SD}(g')$$
$$jg' \mapsto \mathbf{D}(g') \qquad \forall g' \in \mathrm{SO}(3,1).$$

(ii) $$g' \mapsto \mathbf{D}(g') \equiv \left[\begin{array}{c|c} \mathbf{D}(j_0, c; g') & 0 \\ \hline 0 & \mathbf{D}(j_0, -c; g') \end{array}\right] \qquad j_0 \neq 0 \qquad c \neq 0$$
$$s \mapsto \mathbf{S} \equiv \left[\begin{array}{c|c} 0 & \mathbf{S}_1 \\ \hline \mathbf{S}_1 & 0 \end{array}\right]$$
$$t \mapsto -\mathbf{S} \qquad j \mapsto -\mathbf{E} \qquad sg' \mapsto \mathbf{SD}(g') \qquad tg' \mapsto -\mathbf{SD}(g')$$
$$jg' \mapsto -\mathbf{D}(g') \qquad \forall g' \in \mathrm{SO}(3,1).$$

The prescriptions for finding the two-valued irreps of the HLG are contained in the following two theorems (Gelfand *et al* 1963).

Theorem 5. Any pair of non-equivalent mutually conjugate irreps of SO(3, 1) (see theorem 2) may be used to generate a two-valued irrep of the HLG according to the following prescription

$$g' \mapsto \pm\mathbf{D}(g') \equiv \pm\left[\begin{array}{c|c} \mathbf{D}(j_0, c; g') & 0 \\ \hline 0 & \mathbf{D}(j_0, -c; g') \end{array}\right] \qquad j_0 \neq 0 \qquad c \neq 0$$

$$s \mapsto \pm\mathbf{S} \equiv \pm\left[\begin{array}{c|c} 0 & \mathbf{S}_1 \\ \hline \mathbf{S}_1 & 0 \end{array}\right] \qquad t \mapsto \pm\mathbf{T} \equiv \pm\left[\begin{array}{c|c} 0 & \mathrm{i}\mathbf{S}_1 \\ \hline -\mathrm{i}\mathbf{S}_1 & 0 \end{array}\right] \qquad j \mapsto \pm\mathbf{J} \equiv \pm\left[\begin{array}{c|c} -\mathrm{i}\mathbf{E} & 0 \\ \hline 0 & \mathrm{i}\mathbf{E} \end{array}\right]$$

$$sg' \mapsto \pm\mathbf{SD}(g') \qquad tg' \mapsto \pm\mathbf{TD}(g') \qquad jg' \mapsto \pm\mathbf{JD}(g') \qquad \forall g' \in \mathrm{SO}(3,1).$$

Theorem 6. Any self-conjugate irrep of SO(3, 1) *(see theorem 1) may be used to generate* a two-valued irrep of the HLG according to the following prescription

$$g' \mapsto \pm\mathbf{D}(g') \equiv \pm\left[\begin{array}{c|c} \mathbf{D}(j_0, c; g') & 0 \\ \hline 0 & \mathbf{D}(j_0, c; g') \end{array}\right] \qquad \text{either } j_0 = 0 \qquad \text{or } c = 0$$

$$s \mapsto \pm\mathbf{S} \equiv \pm\left[\begin{array}{c|c} \mathbf{S}_1 & 0 \\ \hline 0 & -\mathbf{S}_1 \end{array}\right] \qquad t \mapsto \pm\mathbf{T} \equiv \pm\left[\begin{array}{c|c} 0 & \mathbf{S}_1 \\ \hline \mathbf{S}_1 & 0 \end{array}\right] \qquad j \mapsto \pm\mathbf{J} \equiv \pm\left[\begin{array}{c|c} 0 & \mathbf{E} \\ \hline -\mathbf{E} & 0 \end{array}\right]$$

$$sg' \mapsto \pm\mathbf{SD}(g') \qquad tg' \mapsto \pm\mathbf{TD}(g') \qquad jg' \mapsto \pm\mathbf{JD}(g') \qquad \forall g' \in \mathrm{SO}(3,1).$$

It must be observed that the matrix $\mathbf{S}_1$, that occurs in theorems 2–6, is the same as the $\mathbf{S}_1$ that occurs in theorem 1, and is given by equation (1). Second, the matrix $\mathbf{S}$ that appears in theorems 2, 4, 5 and 6 can be easily checked to possess all the properties of the matrix $\mathbf{S}_1$ quoted in equation (2).

In this context, it may be of some interest to note (see Gelfand *et al* (1963) pp 300–5) that the Dirac equation for the electron, which is well known to be covariant under the decomposable representation $[\mathbf{D}(\frac{1}{2}, \frac{3}{2}) \oplus \mathbf{D}(\frac{1}{2}, -\frac{3}{2})]$ of SO(3, 1), actually transforms according to an irrep of the OLG and a two-valued irrep of the HLG.†

2. The reality classification and some special properties of the irreps of the OLG and the HLG

We now examine the irreps of the OLG and the HLG for a few special properties relating to the bilinear and sesquilinear metrics and C-matrices. For the definition of the C-matrix and the criteria for reality classification of irreps, we refer to two of our earlier papers, namely, parts I and II of this series of three papers on the reality classification of the representations of the HLG (Gopala Rao *et al* 1994a, b).

2.1. *The case of the irreps of the OLG*

Note that although the cases (i) and (ii), occurring in theorem 1, lead to two non-equivalent irreps of the OLG, it is not necessary to consider them separately as the properties of the irreps of the OLG obtained through $s \mapsto +\mathbf{S}_1$ remain valid for the irreps corresponding to $s \mapsto -\mathbf{S}_1$ also. Therefore, in what follows, we consider the irreps corresponding to case (i) only.

Now consider an irrep Δ of the OLG obtained from the self-conjugate irrep $\mathbf{D}(j_0, c)$ of SO(3, 1) from theorem 1. Let $\mathbf{G}_1$ be the (unique) bilinear metric (Srinivasa Rao *et al* 1983) preserved by the irrep $\mathbf{D}(j_0, c)$. Then it is evident that the irrep Δ of the OLG preserves a bilinear metric (in fact $\mathbf{G}_1$ itself) if, and only if, $\tilde{\mathbf{S}}_1\mathbf{G}_1\mathbf{S}_1 = \mathbf{G}_1$, where $\mathbf{S}_1$ is the matrix representative of the space reflection element s (see theorem 1 and equation (1)). The same argument applies with regards to the sesquilinear metric and the C-matrix admitted by Δ. Thus, the results displayed in table 1 follow. In obtaining the results quoted in table 1, we have made use of the available sesquilinear metrics (see Gelfand *et al* (1963) pp 201–7), the bilinear metrics (Srinivasa Rao *et al* 1983) and the C-matrices (Gopala Rao *et al* 1994a, b) of the irreducible and decomposable representations of SO(3, 1).

Consider next, an irrep Δ of the OLG which is obtained from a two-component decomposable representation $[\mathbf{D}(j_0, c) \oplus \mathbf{D}(j_0, -c)]$ from theorem 2. It is evident that if the operator $\mathbf{S}$ in the irrep Δ of the OLG, i.e.

$$\mathbf{S} \equiv \left[\begin{array}{c|c} 0 & \mathbf{S}_1 \\ \hline \mathbf{S}_1 & 0 \end{array}\right]$$

where $\mathbf{S}_1$ is given by equation (1), preserves any one of the various bilinear metrics (Gopala Rao *et al* 1994a) preserved by the decomposable representation $[\mathbf{D}(j_0, c) \oplus \mathbf{D}(j_0, -c)]$, then the corresponding irrep Δ of the OLG also preserves the same bilinear metric. If $\mathbf{S}$ does not preserve any of the bilinear metrics associated with $[\mathbf{D}(j_0, c) \oplus \mathbf{D}(j_0, -c)]$, then Δ does not preserve any bilinear metric at all. Precisely the same arguments hold for the sesquilinear metric and the C-matrix associated with Δ. Thus, we arrive at the results listed in table 2 for the irreps of the OLG (generated from theorem 2).

† In the $\mathbf{D}^{jj'}$ notation for the finite-dimensional irreps of SO(3, 1), this would mean that the Dirac equation transforms according to the $\mathbf{D}^{\frac{1}{2}0} \oplus \mathbf{D}^{0\frac{1}{2}}$ representation.

Table 1. Properties of the irreps of the OLG obtained from theorem 1 and those of the corresponding unique irreps of the HLG obtained from theorem 3.

Case	Parameters characterizing the associated irrep of SO(3, 1)		Properties of the corresponding irreps of OLG and HLG				
no	j_0	c	Bilinear metric $\mathbf{G}_1$	Sesquilinear metric $\mathbf{A}_1$	C-matrix $\mathbf{C}_1$	Reality type	Special properties
1	0	complex	$G_1(j, m; j', m') = (-1)^{-m-j_0}\delta_{j',j}\delta_{m',-m}$	—	—	III	orthogonal
2	0	imaginary	"	$\mathbf{A}_1 = \mathbf{E}$	$\mathbf{C}_1 = \mathbf{G}_1$	I	real orthogonal
3	0	real $0 < \lvert c\rvert \leqslant 1$	"	"	"	I	real orthogonal
4	0	real integer $c \geqslant 2$	"	$\mathbf{A}_1$	$\mathbf{C}_1$	I	real pseudo-orthogonal
5	0	real non-integer $1 < \lvert c\rvert < \infty$	"	$\mathbf{A}_1$	$\mathbf{C}_1$	I	real pseudo-orthogonal
6	integer	0	"	$\mathbf{A}_1 = \mathbf{E}$	$\mathbf{C}_1 = \mathbf{G}_1$	I	real orthogonal
7	half-integer	0	"	"	"	II	unitary symplectic

(i) In case 4, $A_1(j, m; j', m') = (-1)^j\delta_{j',j}\delta_{m',m}$, $C_1(j, m; j', m') = (-1)^{j-m}\delta_{j',j}\delta_{m',-m}$.

(ii) In case 5, $A_1(j, m; j', m') = (-1)^k\delta_{j',j}\delta_{m',m}$, $C_1(j, m; j', m') = (-1)^{k-m}\delta_{j',j}\delta_{m',-m}$, where $k = j$ for $0 \leqslant j \leqslant n$ and $k = n$ for $j > n$, with n denoting the integral part of $\lvert c\rvert$.

(iii) When j_0 is an integer, the corresponding irreps of the OLG are equivalent to orthogonal ones.

(iv) When an irrep of the OLG is potentially-real (i.e., of the first kind), $\mathbf{D}'(g) = \mathbf{T}_1^{-1}\mathbf{D}(g)\mathbf{T}_1 = [\mathbf{D}'(g)]^*(g \in \text{OLG})$, where $\mathbf{T}_1 = [(1 - \mathrm{i})/2][\mathbf{E} + \mathrm{i}\mathbf{C}_1]$.

(v) Note that a representation $\mathbf{D}$ is said to be unitary if there exists a non-degenerate positive-definite matrix $\mathbf{A}$ such that $\mathbf{D}^\dagger(g)\mathbf{A}\mathbf{D}(g) = \mathbf{A}$, $\forall\mathbf{D}(g) \in \mathbf{D}$. Similarly, $\mathbf{D}$ is symplectic if there exists a non-degenerate anti-symmetric matrix $\mathbf{G}$ such that $\tilde{\mathbf{D}}(g)\mathbf{G}\mathbf{D}(g) = \mathbf{G}$, $\forall\mathbf{D}(g) \in \mathbf{D}$. When both an $\mathbf{A}$ and a $\mathbf{G}$ simultaneously exist for a $\mathbf{D}$, it is called a unitary-symplectic representation.

(vi) In all these tables (1–4), the properties of representations listed under the column 'special properties' are to be understood in the sense of equivalence.

Table 2. Properties of the irreps of the OLG obtained from theorem 2 and those of the corresponding unique irreps of the HLG obtained from theorem 4.

Case	Parameters characterizing the associated representation $\mathbf{D}(j_0, c) \oplus \mathbf{D}(j_0, -c)$ of SO(3, 1)		Properties of the corresponding irreps of OLG and HLG				
no	$j_0 \neq 0$	$c \neq 0$	Bilinear metric **G**	Sesquilinear metric **A**	*C*-matrix **C**	Reality type	Special properties
1	integer	complex	$\mathbf{G} = \mathbf{G}_1(j_0, c) \oplus \mathbf{G}_1(j_0, c)$	—	—	III	orthogonal
2	half-integer	complex	"	—	—	III	symplectic
3	integer	imaginary	"	**E**	$\mathbf{C} = \mathbf{G}$	I	real orthogonal
4	half-integer	imaginary	"	"	"	II	unitary symplectic
5	integer	real integer $c^2 > j_0^2$	"	$\mathbf{A} = \mathbf{A}_1 * \mathbf{A}_1$	$\mathbf{C} = \mathbf{C}_1 * \mathbf{C}_1$	I	real pseudo-orthogonal
6	half-integer	real half-integer $c^2 > j_0^2$	"	"	"	II	symplectic
7	integer	real $c^2 < j_0^2$	"	$\mathbf{A} = \mathbf{E} * \mathbf{E}$	$\mathbf{C} = \mathbf{G}_1 * \mathbf{G}_1$	I	real pseudo-orthogonal
8	half-integer	real $c^2 < j_0^2$	"	"	"	II	symplectic
9	integer	real non-integer $c^2 > j_0^2$	"	$\mathbf{A} = \mathbf{A}_1 * \mathbf{A}_1$	$\mathbf{C} = \mathbf{C}_1 * \mathbf{C}_1$	I	real pseudo-orthogonal
10	half-integer	real non-half-integer $c^2 > j_0^2$	"	"	"	II	symplectic

(i) The matrix elements of $\mathbf{G}_1(j_0, c)$ are $\mathbf{G}_1(j, m; j', m') = (-1)^{-m-j_0}\delta_{j',j}\delta_{m',-m}$.
(ii) The notation $\mathbf{A} * \mathbf{B}$ used above stands for the block-matrix

$$\left[\begin{array}{c|c} 0 & \mathbf{A} \\ \hline \mathbf{B} & 0 \end{array}\right].$$

(iii) When j_0 is an integer, the corresponding irreps of the OLG are equivalent to orthogonal irreps.
(iv) In cases 5 and 6, $A_1(j, m; j', m') = (-1)^{j-j_0}\delta_{j',j}\delta_{m',m}$, $C_1(j, m; j', m') = (-1)^{j+m}\delta_{j',j}\delta_{m',-m}$.
(v) In cases 9 to 10,

$$A_1(j, m; j', m') = \begin{cases} (-1)^{j-j_0}\delta_{j',j}\delta_{m',m} & \text{for } j^2 < c^2 \\ (-1)^{2j-j_0-k}\delta_{j',j}\delta_{m',m} & \text{for } j^2 > c^2 \end{cases}$$

$$C_1(j, m; j', m') = \begin{cases} (-1)^{j+m}\delta_{j',j}\delta_{m',-m} & \text{for } j^2 < c^2 \\ (-1)^{2j+m-k}\delta_{j',j}\delta_{m',-m} & \text{for } j^2 > c^2 \end{cases}$$

where k is the integral or half-integral part of c provided that j_0 is an integer or a half-integer.
(vi) When an irrep of the OLG is of the first kind, $\mathbf{D}'(g) = \mathbf{U}^{-1}\mathbf{D}(g)\mathbf{U} = [\mathbf{D}'(g)]^*(g \in \text{OLG})$, where $\mathbf{U} = [(1-\mathrm{i})/2][\mathbf{E} + \mathrm{i}\mathbf{C}]$.

Table 3. Properties of the two-valued irreps of the HLG obtained from theorem 5.

	Parameters characterizing the associated representation $\mathbf{D}(j_0, c) \oplus \mathbf{D}(j_0, -c)$ of SO(3, 1)		Properties of the corresponding two-valued irreps of the HLG			
Case no	$j_0 \neq 0$	$c \neq 0$	Bilinear metric $\mathbf{G}$	Sesquilinear metric $\mathbf{A}$	C-matrix $\mathbf{C}$	Reality type
1	integer or half-integer	complex	—	—	—	III
2	integer or half-integer	imaginary	—	$\mathbf{E}$	—	III
3	integer	real integer $c^2 > j_0^2$	—	—	$\mathbf{C} = \mathbf{C}_1 * \mathbf{C}_1$	I
4	half-integer	real half-integer $c^2 > j_0^2$	—	—	"	II
5	integer	real $c^2 < j_0^2$	—	—	"	I
6	half-integer	real $c^2 < j_0^2$	—	—	"	II
7	integer	real non-integer $c^2 > j_0^2$	—	—	"	I
8	half-integer	real non-half-integer $c^2 > j_0^2$	—	—	"	II

(i) See table 2 for the notation $\mathbf{C}_1 * \mathbf{C}_1$.
(ii) In cases 3 and 4, $\mathbf{C}_1(j, m; j', m') = (-1)^{j+m}\delta_{j',j}\delta_{m',-m}$.
(iii) In cases 5 and 6, $\mathbf{C}_1(j, m; j', m') = (-1)^{-m-j_0}\delta_{j',j}\delta_{m',-m}$.
(iv) In cases 7 and 8 see note (v) concerning table 2 for the elements of $\mathbf{C}_1$.
(v) When j_0 is an integer, the corresponding irrep of the HLG is equivalent to an orthogonal irrep.
(vi) When an irrep of the HLG is potentially-real, $\mathbf{D}'(g) = \mathbf{U}^{-1}\mathbf{D}(g)\mathbf{U} = [\mathbf{D}'(g)]^*(g \in \text{HLG})$, where $\mathbf{U} = [(1-\mathrm{i})/2][\mathbf{E} + \mathrm{i}\mathbf{C}]$.

Table 4. Properties of the two-valued irreps of the HLG obtained from theorem 6.

Case no	Parameters characterizing the associated irrep of SO(3, 1): j_0	c	Properties of the corresponding two-valued irreps of the HLG: Bilinear metric $\mathbf{G}_1$	Sesquilinear metric $\mathbf{A}_1$	C-matrix $\mathbf{C}_1$	Reality type	Special properties
1	0	complex	$\mathbf{G} = \mathbf{G}_1(j_0, c) \oplus \mathbf{G}_1(j_0, c)$	—	—	III	orthogonal
2	0	imaginary	"	$\mathbf{A} = \mathbf{E}$	$\mathbf{C} = \mathbf{G}$	I	real orthogonal
3	0	real $0 < \|c\| \leqslant 1$	"	"	"	I	real orthogonal
4	0	real integer $c \geqslant 2$	"	$\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_1$	$\mathbf{C} = \mathbf{C}_1 \oplus \mathbf{C}_1$	I	real pseudo-orthogonal $2c^2$-dimensional representations
5	0	real non-integer $1 < \|c\| < \infty$	"	"	"	I	real pseudo-orthogonal
6	integer	0	"	$\mathbf{A} = \mathbf{E}$	$\mathbf{C} = \mathbf{G}$	I	real orthogonal
7	half-integer	0	"	"	"	II	unitary symplectic

(i) For the matrix elements of $\mathbf{G}_1(j_0, c)$, see note (i) concerning table 2.
(ii) In case 4, see note (i) concerning table 1 for elements of $\mathbf{A}_1$ and $\mathbf{C}_1$.
(iii) In case 5, see note (ii) concerning table 1 for elements of $\mathbf{A}_1$ and $\mathbf{C}_1$.
(iv) The remark made in note (vi) concerning table 3 also holds here.

2.2. *The case of the unique irreps of the* HLG

Comparing theorems 1 and 3, we observe that precisely the same operators $\mathbf{D}(g')$ and $\mathbf{S}_1$ figure in both these theorems and, as such, every property (displayed in table 1) possessed by a certain irrep of the OLG (obtained from theorem 1) is also true of the corresponding unique irreps of the HLG obtained from theorem 2. Similarly, comparing theorems 2 and 4, we note that all the properties displayed in table 2 of the irreps of the OLG (obtained from theorem 2) are also possessed by the corresponding unique irreps of the HLG obtained from theorem 4.

2.3. *The case of the two-valued irreps of the* HLG

It is easy to see that if all three operators $\mathbf{S}, \mathbf{T}$ and $\mathbf{J}$ which occur in theorem 5 admit any one of the various bilinear metrics (Gopala Rao *et al* 1994b) associated with the decomposable representation $\Delta \equiv [\mathbf{D}(j_0, c) \oplus \mathbf{D}(j_0, -c)]$ in the sense that $\tilde{\mathbf{S}}\mathbf{G}\mathbf{S} = \tilde{\mathbf{T}}\mathbf{G}\mathbf{T} = \tilde{\mathbf{J}}\mathbf{G}\mathbf{J} = \mathbf{G}$, then the two-valued representation of HLG obtained from theorem 5 also admits the same bilinear metric $\mathbf{G}$. Conversely, if these three operators $\mathbf{S}, \mathbf{T}$ and $\mathbf{J}$ do not admit any of the bilinear *metrics admitted by the representation* Δ *of* SO(3, 1), *then the corresponding two-valued* irrep of the HLG (obtained from theorem 5) clearly does not admit any bilinear metric. Similar arguments hold with regard to the sesquilinear metric and the C-matrix associated with the two-valued irreps of the HLG obtained from theorem 5. These observations lead to a number of metric and reality properties of the two-valued irreps of the HLG generated from theorem 5, and these results are displayed in table 3.

Similar arguments lead us to the properties listed in table 4 for the two-valued irreps of the HLG, obtained from theorem 6.

Acknowledgment

We are grateful to our teacher Professor K N Srinivasa Rao for helpful discussions and encouragement throughout the preparation of this paper.

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